



Oxford Cambridge and RSA

Monday 19 October 2020 – Afternoon

A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension

Time allowed: 2 hours



You must have:

- the Printed Answer Booklet
- the Insert
- a scientific or graphical calculator

INSTRUCTIONS

- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION

- The total mark for this paper is **75**.
- The marks for each question are shown in brackets [].
- This document has **12** pages.

ADVICE

- Read each question carefully before you start your answer.

Formulae A Level Mathematics B (MEI) (H640)

Arithmetic series

$$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n\{2a + (n-1)d\}$$

Geometric series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \text{ for } |r| < 1$$

Binomial series

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \dots + {}^nC_r a^{n-r}b^r + \dots + b^n \quad (n \in \mathbb{N}),$$

$$\text{where } {}^nC_r = {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!}x^r + \dots \quad (|x| < 1, n \in \mathbb{R})$$

Differentiation

$f(x)$	$f'(x)$
$\tan kx$	$k \sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

$$\text{Quotient Rule } y = \frac{u}{v}, \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

Differentiation from first principles

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Integration

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$$

$$\int f'(x)(f(x))^n dx = \frac{1}{n+1}(f(x))^{n+1} + c$$

$$\text{Integration by parts } \int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

Small angle approximations

$$\sin \theta \approx \theta, \cos \theta \approx 1 - \frac{1}{2}\theta^2, \tan \theta \approx \theta \text{ where } \theta \text{ is measured in radians}$$

Trigonometric identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad \left(A \pm B \neq \left(k + \frac{1}{2}\right)\pi \right)$$

Numerical methods

Trapezium rule: $\int_a^b y \, dx \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})\}$, where $h = \frac{b-a}{n}$

The Newton-Raphson iteration for solving $f(x) = 0$: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B) \quad \text{or} \quad P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Sample variance

$$s^2 = \frac{1}{n-1}S_{xx} \quad \text{where} \quad S_{xx} = \sum(x_i - \bar{x})^2 = \sum x_i^2 - \frac{(\sum x_i)^2}{n} = \sum x_i^2 - n\bar{x}^2$$

Standard deviation, $s = \sqrt{\text{variance}}$

The binomial distribution

If $X \sim B(n, p)$ then $P(X = r) = {}^n C_r p^r q^{n-r}$ where $q = 1 - p$

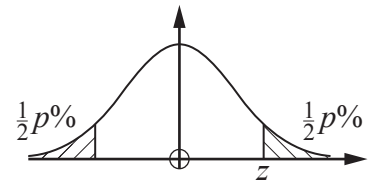
Mean of X is np

Hypothesis testing for the mean of a Normal distribution

If $X \sim N(\mu, \sigma^2)$ then $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ and $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$

Percentage points of the Normal distribution

p	10	5	2	1
z	1.645	1.960	2.326	2.576

**Kinematics**

Motion in a straight line

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$$s = \frac{1}{2}(u + v)t$$

$$v^2 = u^2 + 2as$$

$$s = vt - \frac{1}{2}at^2$$

Motion in two dimensions

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$$

$$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$$

Answer **all** the questions.

Section A (60 marks)

1 Find the value of $\sum_{r=1}^5 2^r(r-1)$. [2]

2 The graph of $y = |1-x|-2$ is shown in Fig. 2.

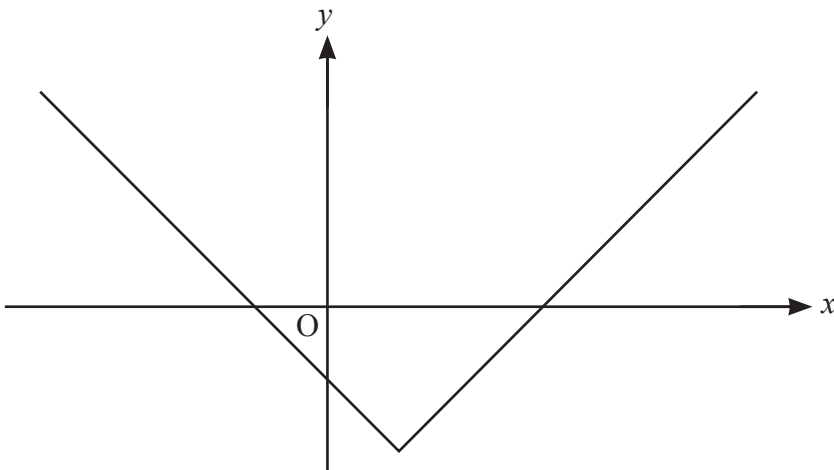


Fig. 2

Determine the set of values of x for which $|1-x| > 2$. [4]

3 A particular phone battery will last 10 hours when it is first used. Every time it is recharged, it will only last 98% of its previous time.

Find the maximum total length of use for the battery. [3]

4 Fig. 4 shows the regular octagon ABCDEFGH.

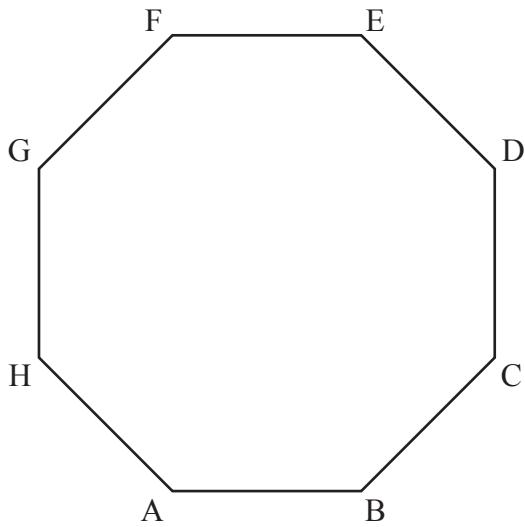


Fig. 4

$\overrightarrow{AB} = \mathbf{i}$, $\overrightarrow{CD} = \mathbf{j}$, where \mathbf{i} is a unit vector parallel to the x -axis and \mathbf{j} is a unit vector parallel to the y -axis.

Find an exact expression for \overrightarrow{BC} in terms of \mathbf{i} and \mathbf{j} .

[3]

- 5 Fig. 5 shows part of the curve $y = \operatorname{cosec} x$ together with the x - and y -axes.

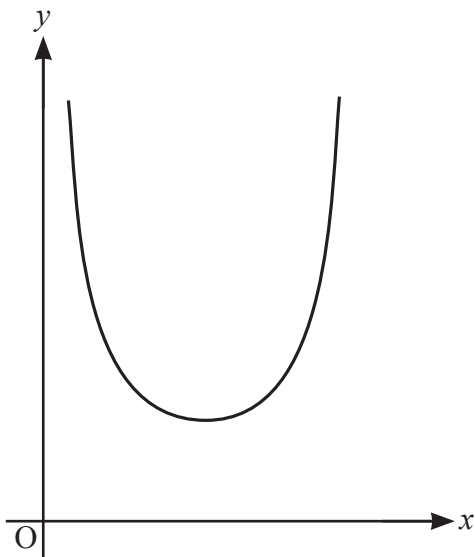


Fig. 5

- (a) For the section of the curve which is shown in Fig. 5, write down
- (i) the equations of the two vertical asymptotes, [2]
 - (ii) the coordinates of the minimum point. [1]
- (b) Show that the equation $x = \operatorname{cosec} x$ has a root which lies between $x = 1$ and $x = 2$. [2]
- (c) Use the iteration $x_{n+1} = \operatorname{cosec}(x_n)$, with $x_0 = 1$, to find
- (i) the values of x_1 and x_2 , correct to 5 decimal places, [1]
 - (ii) this root of the equation, correct to 3 decimal places. [1]
- (d) There is another root of $x = \operatorname{cosec} x$ which lies between $x = 2$ and $x = 3$.
- Determine whether the iteration $x_{n+1} = \operatorname{cosec}(x_n)$ with $x_0 = 2.5$ converges to this root. [1]
- (e) Sketch the staircase or cobweb diagram for the iteration, starting with $x_0 = 2.5$, on the diagram in the Printed Answer Booklet. [3]

- 6 (a) (i) Write down the derivative of e^{kx} , where k is a constant. [1]
- (ii) A business has been running since 2009. They sell maths revision resources online.

Give a reason why an exponential growth model might be suitable for the annual profits for the business. [1]

Fig. 6 shows the relationship between the annual profits of the business in thousands of pounds (y) and the time in years after 2009 (x). The graph of $\ln y$ plotted against x is approximately a straight line.

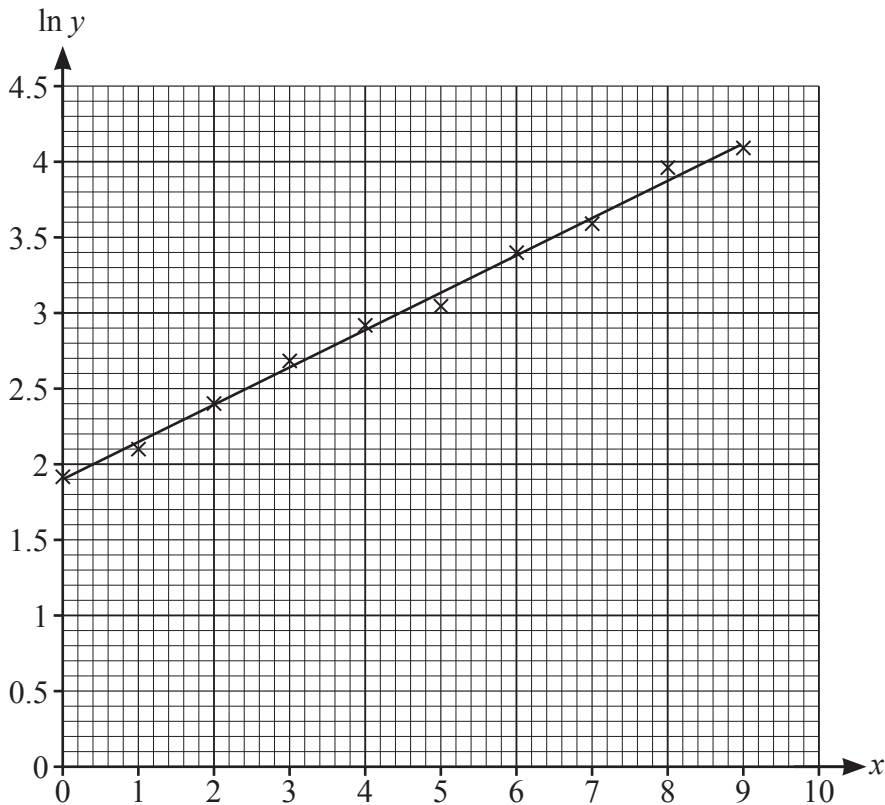


Fig. 6

- (b) Show that the straight line is consistent with a model of the form $y = Ae^{kx}$, where A and k are constants. [2]
- (c) Estimate the values of A and k . [4]
- (d) Use the model to predict the profit in the year 2020. [3]
- (e) How reliable do you expect the prediction in part (d) to be? Justify your answer. [1]

- 7 (a) Express $\frac{1}{x} + \frac{1}{A-x}$ as a single fraction. [1]

The population of fish in a lake is modelled by the differential equation

$$\frac{dx}{dt} = \frac{x(400-x)}{400}$$

where x is the number of fish and t is the time in years.

When $t = 0$, $x = 100$.

- (b) In this question you must show detailed reasoning.

Find the number of fish in the lake when $t = 10$, as predicted by the model. [8]

- 8 (a) The curve $y = \frac{1}{(1+x^2)^2}$ is shown in Fig. 8.

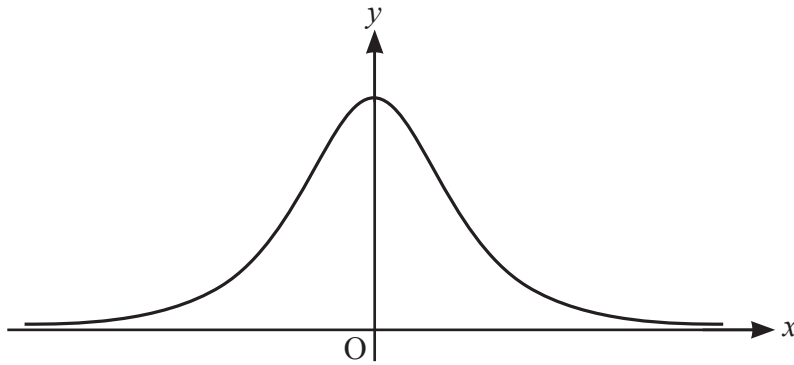


Fig. 8

- (i) Show that $\frac{d^2y}{dx^2} = \frac{20x^2 - 4}{(1+x^2)^4}$. [5]

- (ii) In this question you must show detailed reasoning.

Find the set of values of x for which the curve is concave downwards. [3]

- (b) Use the substitution $x = \tan \theta$ to find the exact value of $\int_{-1}^1 \frac{1}{(1+x^2)^2} dx$. [8]

Answer **all** the questions.

Section B (15 marks)

The questions in this section refer to the article on the Insert. You should read the article before attempting the questions.

9 (a) Show that if $a = 1$ and $b > 1$ then $a^b < b^a$. [2]

(b) Find integer values of a and b with $b > a > 1$ and a^b not greater than b^a (a counter example to the conjecture given in lines 7–8). [1]

10 In this question you must show detailed reasoning.

Show that $\int_e^\pi \frac{1}{x} dx = \ln \pi - 1$ as given in line 37. [2]

11 Show that e^x is an increasing function for all values of x , as stated in line 39. [2]

12 (a) Show that the only stationary point on the curve $y = \frac{\ln x}{x}$ occurs where $x = e$, as given in line 45. [3]

(b) Show that the stationary point is a maximum. [3]

(c) It follows from part (b) that, for any positive number a with $a \neq e$,

$$\frac{\ln e}{e} > \frac{\ln a}{a}.$$

Use this fact to show that $e^a > a^e$. [2]

END OF QUESTION PAPER

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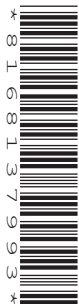
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INSTRUCTIONS

- Do **not** send this Insert for marking. Keep it in the centre or recycle it.

INFORMATION

- This Insert contains the article for Section B.
- This document has **4** pages.

Which is bigger?

Which is bigger: π^e or e^π ? Using a calculator confirms that e^π is the larger, but how can this be proved without the use of a calculator?

Simpler examples

It is often helpful in mathematics to consider simpler examples. It is easy to work out that $3^4 > 4^3$. In the expression 3^4 , 3 is the base and 4 is the exponent. Working with integers greater than 1, it is easy to find many examples where $a^b > b^a$ if $a < b$. That is, using the smaller base and the larger exponent gives the larger result. This might lead us to conjecture that $a^b > b^a$ if $a < b$ and both a and b are integers greater than 1. However, it is also possible to find counter examples to this conjecture. 5

Exponents can also be rational numbers, and in general $x^{\frac{p}{q}}$ denotes $(\sqrt[q]{x})^p$ where p and q are integers and q is positive. So, any rational power of a positive number, x , can be defined. However, both e and π are irrational numbers. Considering the original question about π^e and e^π raises the issue of what is meant by an irrational power of a number. 10

Extending the definition of power to irrational numbers

What, for example, is meant by 2^π ? 15

An irrational number corresponds to a non-recurring infinite decimal. Rounding the decimal gives a rational approximation to the irrational number. For example, the following sequence gives increasingly accurate approximations to π .

3, 3.1, 3.14, 3.142, 3.1416, 3.14159, ...

Using a spreadsheet gives a sequence of approximations to 2^π , as shown in Fig. C1. The limit of this sequence of approximations is the value of 2^π . This limit cannot be evaluated with a spreadsheet but it is, in principle, possible to find the value to any required degree of accuracy. 20

	A	B
1	k	2^k
2	3	8
3	3.1	8.574188
4	3.14	8.815241
5	3.142	8.82747
6	3.1416	8.825023
7	3.14159	8.824962

Fig. C1

2^x and x^2 are increasing functions of x for $x > 0$ and this allows us to deduce that $\pi^2 > 2^\pi$, as follows.

We know that π is between 3 and 3.142

25

$$\pi < 3.142 \Rightarrow 2^\pi < 2^{3.142} = 8.82747$$

$$\pi > 3 \Rightarrow \pi^2 > 3^2 = 9$$

$$\text{So } \pi^2 > 9 > 8.82747 > 2^\pi$$

$$\text{Hence } \pi^2 > 2^\pi$$

Which is bigger: π^e or e^π ?

30

An indirect method, using calculus, enables us to prove that e^π is larger than π^e . Fig. C2 shows the curve $y = \frac{1}{x}$ in the first quadrant together with the rectangle with vertices at the points $(e, 0)$, $(e, \frac{1}{e})$, $(\pi, \frac{1}{e})$ and $(\pi, 0)$. We use the fact that the area under the curve between e and π is less than the area of this rectangle.

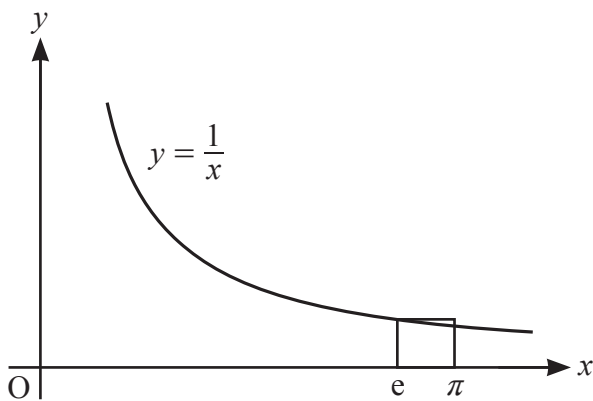


Fig. C2

The area of the rectangle is $\frac{1}{e}(\pi - e)$

35

$$\int_e^\pi \frac{1}{x} dx < \frac{1}{e}(\pi - e)$$

$$\ln \pi - 1 < \frac{\pi}{e} - 1$$

$$\ln \pi < \frac{\pi}{e}$$

e^x is an increasing function for all values of x

$$\text{hence } \pi < e^{\frac{\pi}{e}}$$

40

Assuming that the usual rules of indices apply to irrational powers of irrational numbers, raising both sides of the inequality to the power e gives the desired result.

Using a similar method, it can be shown that $e^a > a^e$ for any positive number $a \neq e$.

An alternative method for showing that $e^a > a^e$ for any positive number a is to show that the only stationary point on the curve $y = \frac{\ln x}{x}$ (a maximum) occurs where $x = e$.

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