

Mark Scheme (Post-Standardisation) Summer 2007

GCE

GCE Mathematics (6676/01)

June 2007
6676 Further Pure Mathematics FP3
Mark Scheme

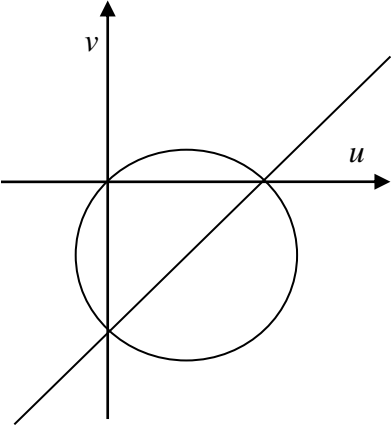
Question Number	Scheme	Marks
1.	<p>(a) $\frac{y_1 - 0.2}{0.1} \approx \left(\frac{dy}{dx}\right)_0 = 0.2 \times e^0 \quad (= 0.2)$ $y_1 \approx 0.22$</p> <p>(b) $\left(\frac{dy}{dx}\right)_1 \approx 0.22 \times e^{0.01} \approx 0.2222 \dots$ $\frac{y_2 - 0.2}{0.2} \approx 0.2222 \dots$ $y_2 \approx 0.2444$</p>	<p>M1 A1 (2)</p> <p>B1 M1 A1 (3) [5]</p>
2.	<p>(a) $(1 - x^2) \frac{d^3y}{dx^3} - 2x \frac{d^2y}{dx^2} - x \frac{d^2y}{dx^2} - \frac{dy}{dx} + 2 \frac{dy}{dx} = 0$ At $x = 0$, $\frac{d^3y}{dx^3} = -\frac{dy}{dx} = 1$</p> <p>(b) $\left(\frac{d^2y}{dx^2}\right)_0 = -4$ Allow anywhere $y = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \dots$ $= 2 - x - 2x^2 + \frac{1}{6}x^3 + \dots$</p>	<p>M1 M1 A1 cso (3)</p> <p>B1 M1 A1ft, A1 (dep)(4) [7]</p>

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3.	<p>(a) $\begin{pmatrix} 3 & 4 & p \\ -1 & q & -4 \\ 1 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$</p> <p>Third row $1 - 3 = -\lambda \Rightarrow \lambda = 2$</p>	M1 A1 (2)
	<p>(b) $\begin{pmatrix} 3 & 4 & p \\ -1 & q & -4 \\ 1 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 - p \\ q + 4 \\ -2 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$</p> <p>First row $4 - p = 0 \Rightarrow p = 4$ Method for either</p> <p>Second row $q + 4 = 2 \Rightarrow q = -2$ Both correct</p>	M1 A1 M1 A1 ft (4)
	<p>(c) $\begin{pmatrix} 3 & 4 & 4 \\ -1 & -2 & -4 \\ 1 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix}$</p> <p>$3l + 4m + 4n = 10$ $-l - 2m - 4n = -4$ $l + m + 3n = 3$ Obtaining 3 linear equations</p> <p>$2l + 2m = 6$ $3l + 2m = 8$ Reducing to a pair of equations and solving for one variable</p> <p>$l = 2, m = 1, n = 0$ Solving for all three variables.</p>	M1 M1 M1 A1 (4) [10]
	<p>Alternative to (c)</p> <p>$\mathbf{A}^{-1} = \frac{1}{6} \begin{pmatrix} 2 & 8 & 8 \\ -1 & -5 & -8 \\ -1 & -1 & 2 \end{pmatrix}$</p> <p>$\frac{1}{6} \begin{pmatrix} 2 & 8 & 8 \\ -1 & -5 & -8 \\ -1 & -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$</p>	M1 M1 M1 A1 (4)

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4.	<p>(a) $z^n = (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$ $z^{-n} = (\cos \theta + i \sin \theta)^{-n} = \cos(-n\theta) + i \sin(-n\theta) = \cos n\theta - i \sin n\theta$ both</p> <p>Adding $z^n + \frac{1}{z^n} = 2 \cos n\theta$ * cso</p> <p>(b) $\left(z + \frac{1}{z}\right)^6 = z^6 + 6z^4 + 15z^2 + 20 + 15z^{-2} + 6z^{-4} + z^{-6}$ M1 $= z^6 + z^{-6} + 6(z^4 + z^{-4}) + 15(z^2 + z^{-2}) + 20$ M1 $64 \cos^6 \theta = 2 \cos 6\theta + 12 \cos 4\theta + 30 \cos 2\theta + 20$ M1 $32 \cos^6 \theta = \cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10$ A1, A1 <p style="text-align: center;">($p = 1, q = 6, r = 15, s = 10$) A1 any two correct (5)</p> <p>(c) $\int \cos^6 \theta d\theta = \left(\frac{1}{32}\right) \int (\cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10) d\theta$ $= \left(\frac{1}{32}\right) \left[\frac{\sin 6\theta}{6} + \frac{6 \sin 4\theta}{4} + \frac{15 \sin 2\theta}{2} + 10\theta \right]$ M1 A1ft $\left[\dots \right]_0^{\frac{\pi}{3}} = \frac{1}{32} \left[-\frac{3}{2} \times \frac{\sqrt{3}}{2} + \frac{15}{2} \times \frac{\sqrt{3}}{2} + \frac{10\pi}{3} \right] = \frac{5\pi}{48} + \frac{3\sqrt{3}}{32}$ or exact equivalent M1 A1 (4)</p> </p>	<p>(2)</p> <p>(5)</p> <p>(4)</p> <p>[11]</p>

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5.	<p>$n = 1:$ $1^2 = \frac{1}{3} \times 1 \times 1 \times 3$</p> <p>(Hence result is true for $n = 1$.)</p> $\sum_{r=1}^{k+1} (2r-1)^2 = \sum_{r=1}^k (2r-1)^2 + (2k+1)^2$ $= \frac{1}{3} k(2k-1)(2k+1) + (2k+1)^2, \text{ by induction hypothesis}$ $= \frac{1}{3} (2k+1)(2k^2 - k + 6k + 3)$ $= \frac{1}{3} (2k+1)(2k^2 + 5k + 3)$ $= \frac{1}{3} (2k+1)(2k+3)(k+1)$ $= \frac{1}{3} (k+1)[2(k+1)-1][2(k+1)+1]$ <p>(Hence, if result is true for $n = k$, then it is true for $n = k + 1$.)</p> <p>By Mathematical Induction, above implies the result is true for all $n \in \mathbb{N}^+.$ *</p> <p style="text-align: right;">cso</p>	<p>B1</p> <p>M1</p> <p>M1 A1</p> <p>A1 (5) [5]</p>
6.	<p>(a) $f(k+1) - f(k) = 3^{4k+4} + 2^{4k+6} - 3^{4k} - 2^{4k+2}$</p> $= 3^{4k}(3^4 - 1) + 2^{4k+2}(2^4 - 1)$ $= 3^{4k} \times 80 + 2^{4k+2} \times 15$ <p style="text-align: right;">can be implied</p> $= 3^{4k-1} \times 240 + 2^{4k+2} \times 15 = 15(16 \times 3^{4k-1} + 2^{4k+2})$ <p>Hence $15 \mid f(k+1) - f(k)$ *</p> <p style="text-align: right;">cso</p> <p>Note: $f(k+1) - f(k)$ is divisible by 240 and other appropriate multiples of 15 lead to the required result.</p> <p>(b) $n = 1:$ $f(1) = 3^4 + 2^6 = 145 = 5 \times 29 \Rightarrow 5 \mid f(1)$</p> <p>(Hence result is true for $n = 1$.)</p> <p>From (a) $f(k+1) - f(k) = 15\lambda$, say. By induction hypothesis $f(k) = 5\mu$, say.</p> $f(k+1) = f(k) + 15\lambda = 5(\mu + 3\lambda) \Rightarrow 5 \mid f(k+1)$ <p>(Hence, if result is true for $n = k$, then it is true for $n = k + 1$.)</p> <p>By Mathematical Induction, above implies the result is true for all $n \in \mathbb{N}^+.$ *</p> <p>Accept equivalent arguments</p> <p style="text-align: right;">cso</p> <p>(c) $f(1) = 145 = 5 \times 29$ is not divisible by 15, so result is not true for all $\mathbb{N}^+.$</p> <p>Note: There is no integer for which $f(n)$ is divisible by 15 and any specific example should be accepted.</p>	<p>M1</p> <p>A1</p> <p>M1</p> <p>A1 (4)</p> <p>B1</p> <p>M1</p> <p>A1 (3)</p> <p>B1 (1) [8]</p>

Question Number	Scheme	Marks
7.	<p>(a) $\overrightarrow{AB} = -\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$, $\overrightarrow{AC} = 4\mathbf{j} + 2\mathbf{k}$ any two</p> $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 3 \\ 0 & 4 & 2 \end{vmatrix} = -6\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ <p>Give A1 for any two components correct or the negative of the correct answer. (4)</p> <p>(b) Cartesian equation has form $3x - y + 2z = p$ $(2, -1, 0) \Rightarrow 6 + 1 = p$ or use of another point M1 $3x - y + 2z = 7$ * or any multiple A1 (2)</p> <p>(c) Parametric form of line is $\mathbf{r} = \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ or equivalent form M1 A1</p> <p>Substituting into equation of plane $3(5 + 2\lambda) - (5 - \lambda) + 2(3 - 2\lambda) = 7$ M1 Leading to $\lambda = -3$ A1 $T : (-1, 8, 9)$ A1 (5)</p> <p>(d) $\overrightarrow{AT} = -3\mathbf{i} + 9\mathbf{j} + 9\mathbf{k}$, $\overrightarrow{BT} = -2\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}$ both M1 These are parallel and hence A, B and T are collinear * (by the axiom of parallels) M1 A1 (3) [14]</p> <p>Alternative to (d) The equation of AB: $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mu(-\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$ or equivalent \mathbf{i}: $-1 = 2 - \mu \Rightarrow \mu = 3$ M1 $\mu = 3 \Rightarrow \overrightarrow{OT} = -\mathbf{i} + 8\mathbf{j} + 9\mathbf{k}$ M1 Hence A, B and T are collinear * cso A1 (3)</p> <p>Note: Column vectors or bold-faced vectors may be used at any stage.</p>	

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8.	<p>(a) Let $z = \lambda + \lambda i$; $w = \frac{\lambda + (\lambda + 1)i}{\lambda(1 + i)}$</p> $= \frac{\lambda + (\lambda + 1)i}{\lambda(1 + i)} \times \frac{1 - i}{1 - i}$ $u + iv = \frac{(2\lambda + 1) + i}{2\lambda}$ $u = 1 + \frac{1}{2\lambda}, \quad v = \frac{1}{2\lambda}$ <p>Eliminating λ gives a line with equation $v = u - 1$</p>	<p>M1</p> <p>M1</p> <p>A1</p> <p>M1</p> <p>or equivalent A1 (5)</p>
	<p>(b) Let $z = \lambda - (\lambda + 1)i$; $w = \frac{\lambda - \lambda i}{\lambda - (\lambda + 1)i}$</p> $= \frac{\lambda - \lambda i}{\lambda - (\lambda + 1)i} \times \frac{\lambda + (\lambda + 1)i}{\lambda + (\lambda + 1)i}$ $u + iv = \frac{\lambda(2\lambda + 1) + \lambda i}{2\lambda^2 + 2\lambda + 1}$ $u = \frac{\lambda(2\lambda + 1)}{2\lambda^2 + 2\lambda + 1}, \quad v = \frac{\lambda}{2\lambda^2 + 2\lambda + 1}$ $\frac{u}{v} = 2\lambda + 1$ $v = \frac{2\lambda}{4\lambda^2 + 4\lambda + 2} = \frac{(2\lambda + 1) - 1}{(2\lambda + 1)^2 + 1} = \frac{\frac{u}{v} - 1}{\left(\frac{u}{v}\right)^2 + 1}$ <p>Reducing to the circle with equation $u^2 + v^2 - u + v = 0$ *</p>	<p>M1</p> <p>M1</p> <p>A1</p> <p>M1</p> <p>M1</p> <p>cs0 M1 A1 (7)</p>
	<p>(c)</p>  <p>Circle through origin, centre in correct quadrant Intersections correctly placed</p>	<p>ft their line B1ft B1 B1 (3) [15]</p>

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8.	<p>Alternative for (b)</p> <p>Let $z = \lambda - (\lambda + 1)i$; $w = \frac{\lambda - \lambda i}{\lambda - (\lambda + 1)i}$</p> $= \frac{\lambda - \lambda i}{\lambda - (\lambda + 1)i} \times \frac{\lambda + (\lambda + 1)i}{\lambda + (\lambda + 1)i}$ $u + iv = \frac{\lambda(2\lambda + 1) + \lambda i}{2\lambda^2 + 2\lambda + 1}$ $u = \frac{\lambda(2\lambda + 1)}{2\lambda^2 + 2\lambda + 1}, \quad v = \frac{\lambda}{2\lambda^2 + 2\lambda + 1}$ $u^2 + v^2 - u + v = \left(\frac{\lambda(2\lambda + 1)}{2\lambda^2 + 2\lambda + 1}\right)^2 + \left(\frac{\lambda}{2\lambda^2 + 2\lambda + 1}\right)^2 - \frac{\lambda(2\lambda + 1)}{2\lambda^2 + 2\lambda + 1} + \frac{\lambda}{2\lambda^2 + 2\lambda + 1}$ $= \frac{(4\lambda^4 + 4\lambda^3 + \lambda^2) + \lambda^2 - 2\lambda^2(2\lambda^2 + 2\lambda + 1)}{(2\lambda^2 + 2\lambda + 1)^2}$ $= 0 *$	<p>M1</p> <p>M1</p> <p>A1</p> <p>M1</p> <p>M1</p> <p>M1 A1 (7)</p>
8.	<p>Alternative for (b)</p> <p>Let $z = \lambda - (\lambda + 1)i$; $u + iv = \frac{\lambda - \lambda i}{\lambda - (\lambda + 1)i}$</p> $(u + iv)(\lambda - (\lambda + 1)i) = \lambda - \lambda i$ $u\lambda + v(\lambda + 1) + [v\lambda - u(\lambda + 1)]i = \lambda - \lambda i$ <p>Equating real & imaginary parts</p> $u\lambda + v(\lambda + 1) = \lambda \quad \text{(i)} \quad v\lambda - \lambda u - u = -\lambda \quad \text{(ii)}$ <p>From (i) $\lambda = \frac{v}{1 - u - v}$ From (ii) $\lambda = \frac{u}{1 - u + v}$</p> $\frac{v}{1 - u - v} = \frac{u}{1 - u + v}$ <p>Reducing to the circle with equation $u^2 + v^2 - u + v = 0 *$</p>	<p>M1</p> <p>M1</p> <p>A1</p> <p>M1</p> <p>M1</p> <p>M1 A1 (7)</p>