

GCE

# **Mathematics**

**Advanced GCE** 

Unit 4727: Further Pure Mathematics 3

## Mark Scheme for June 2011

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1 (i)	$\theta = \sin^{-1} \frac{\left[ [5, 6, -7] \cdot [1, 2, -1] \right]}{\sqrt{5^2 + 6^2 + (-7)^2} \sqrt{1^2 + 2^2 + (-1)^2}}$	M1*	For using scalar product of line and plane vectors For both moduli seen
	$\sqrt{3} + 6 + (-1) \sqrt{1} + 2 + (-1)$	(*dep)	
	$\theta = \sin^{-1} \frac{24}{\sqrt{110}\sqrt{6}} = 69.1^{\circ} (69.099^{\circ}, 1.206)$	A1 A1 4	For correct scalar product For correct angle
		SR	For vector product of line and plane vectors
	$\phi = \sin^{-1} \frac{\left  [5, 6, -7] \times [1, 2, -1] \right }{\sqrt{5^2 + 6^2 + (-7)^2} \sqrt{1^2 + 2^2 + (-1)^2}}$	M1*	AND finding modulus of result
	<u> </u>	M1 (*dep)	For moduli of line and plane vectors seen
	$\phi = \sin^{-1} \frac{\sqrt{84}}{\sqrt{110}\sqrt{6}} = 20.9^{\circ} \implies \theta = 69.1^{\circ}$	A1 A1	For correct modulus $\sqrt{84}$
	γ110γ0	A1	For correct angle
(ii)	METHOD 1	M1	Far was of someof formalls
	$d = \frac{ 1+12+3-40 }{\sqrt{1^2+2^2+(-1)^2}} = \frac{24}{\sqrt{6}} = 4\sqrt{6} \approx 9.80$	M1 A1 <b>2</b>	For use of correct formula For correct distance
	V1 12 1 ( 1)	A1 2	1 of correct distance
	METHOD 2		
	$(1+\lambda) + 2(6+2\lambda) - (-3-\lambda) = 40$	M1	For substituting parametric form into plane
	$\Rightarrow \lambda = 4 \Rightarrow d = 4\sqrt{6}$	<b>A</b> 1	For correct distance
	<i>OR</i> distance from $(1, 6, -3)$ to $(5, 14, -7)$		
	$=\sqrt{4^2+8^2+(-4)^2}=\sqrt{96}$		
	METHOD 3		
	Plane through $(1, 6, -3)$ parallel to $p$ is	M1	For finding parallel plane through $(1, 6, -3)$
	$x + 2y - z = 16 \implies d = \frac{40 - 16}{\sqrt{6}} = \frac{24}{\sqrt{6}}$	A1	For correct distance
	METHOD 4		
	e.g. $(0, 0, -40)$ on $p$	M1	For using any point on p to find vector
	$\Rightarrow$ vector to $(1, 6, -3) = \pm (1, 6, 37)$		and scalar product seen e.g. [1, 6, 37] • [1, 2, -1]
	$d = \frac{ [1, 6, 37] \cdot [1, 2, -1] }{\sqrt{6}} = \frac{24}{\sqrt{6}}$	A1	For correct distance
	METHOD 5		
	<i>l</i> meets <i>p</i> where $(1+5t)+2(6+6t)-(-3-7t)=40$		For finding $t$ where $l$ meets $p$
	$\Rightarrow t = 1 \Rightarrow d =  [5, 6, -7]  \sin \theta$	M1	and linking $d$ with triangle
	$\Rightarrow d = \sqrt{110} \frac{24}{\sqrt{110}\sqrt{6}} = \frac{24}{\sqrt{6}}$	A1	For correct distance
	√110√6 √6	6	
2 (3)	METHOD 1		.1:0
2 (i)	1 1	M1	EITHER For changing LHS terms to $e^{\pm \frac{1}{2}i\theta}$
	EITHER $\frac{1+e^{i\theta}}{1-e^{i\theta}} = \frac{e^{-\frac{1}{2}i\theta} + e^{\frac{1}{2}i\theta}}{e^{-\frac{1}{2}i\theta} - e^{\frac{1}{2}i\theta}}$		OR in reverse For using $\cot \frac{1}{2}\theta = \frac{\cos \frac{1}{2}\theta}{\sin \frac{1}{2}\theta}$
	$= \frac{2\cos\frac{1}{2}\theta}{-2i\sin\frac{1}{2}\theta} = i\cot\frac{1}{2}\theta$	M1	
	$-2i\sin\frac{1}{2}\theta$		For either of $\frac{\cos \frac{1}{2}\theta}{\sin \frac{1}{2}\theta} = \frac{e^{\frac{1}{2}i\theta} \pm e^{-\frac{1}{2}i\theta}}{(2)(i)}$ soi
	OR in reverse with similar working	A1 3	For fully correct proof to <b>AG SR</b> If factors of 2 or i are not clearly seen,
			award M1 M1 A0

#### METHOD 2 2 (i)

$$\textit{EITHER} \ \frac{1 + e^{i\theta}}{1 - e^{i\theta}} \times \frac{1 - e^{-i\theta}}{1 - e^{-i\theta}} = \frac{e^{i\theta} - e^{-i\theta}}{2 - \left(e^{i\theta} + e^{-i\theta}\right)}$$

For multiplying top and bottom by complex M1 conjugate in exp or trig form

$$OR \frac{1 + \cos\theta + i\sin\theta}{1 - \cos\theta - i\sin\theta} \times \frac{1 - \cos\theta + i\sin\theta}{1 - \cos\theta + i\sin\theta}$$

$$= \frac{2i\sin\theta}{2 - 2\cos\theta} = \frac{2i\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta} = i\cot\frac{1}{2}\theta$$

M1 For using both double angle formulae correctly

#### Α1 For fully correct proof to AG

#### METHOD 3

$$\frac{1+\cos\theta+\mathrm{i}\sin\theta}{1-\cos\theta-\mathrm{i}\sin\theta} = \frac{2\cos^2\frac{1}{2}\theta+2\mathrm{i}\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta-2\mathrm{i}\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}$$

M1 For using both double angle formulae correctly

$$=\frac{2\cos\frac{1}{2}\theta\left(\cos\frac{1}{2}\theta+i\sin\frac{1}{2}\theta\right)}{2\sin\frac{1}{2}\theta\left(\sin\frac{1}{2}\theta-i\cos\frac{1}{2}\theta\right)}$$

M1 For appropriate factorisation

$$= i\cot\frac{1}{2}\theta \frac{\left(\sin\frac{1}{2}\theta - i\cos\frac{1}{2}\theta\right)}{\left(\sin\frac{1}{2}\theta - i\cos\frac{1}{2}\theta\right)} = i\cot\frac{1}{2}\theta$$

**A**1 For fully correct proof to AG

#### METHOD 4

$$\frac{1 + \cos \theta + i \sin \theta}{1 - \cos \theta - i \sin \theta} = \frac{1 + \frac{1 - t^2}{1 + t^2} + i \frac{2t}{1 + t^2}}{1 - \frac{1 - t^2}{1 + t^2} - i \frac{2t}{1 + t^2}}$$

M1 For substituting both t formulae correctly

$$= \frac{2+2it}{2t^2-2it} = \frac{1}{t} \frac{1+it}{t-i} = \frac{i}{t} \frac{t-i}{t-i} = i \cot \frac{1}{2}\theta$$

For appropriate factorisation M1 **A**1 For fully correct proof to AG

### METHOD 5

$$\begin{aligned} &\frac{1+e^{i\theta}}{1-e^{i\theta}} \times \frac{1+e^{i\theta}}{1+e^{i\theta}} = \frac{1+2e^{i\theta}+e^{2i\theta}}{1-e^{2i\theta}} \\ &= \frac{2+e^{i\theta}+e^{-i\theta}}{e^{-i\theta}-e^{i\theta}} \end{aligned}$$

For multiplying top and bottom by  $1 + e^{i\theta}$ 

and attempting to divide by  $e^{i\theta}$ M1 *OR* multiplying top and bottom by  $1 + e^{-i\theta}$ 

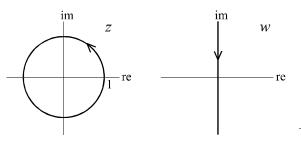
 $=\frac{2(1+\cos\theta)}{-2\sin\theta}=\frac{2\cos^2\frac{1}{2}\theta}{-2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}=\frac{\cos\frac{1}{2}\theta}{-\sin\frac{1}{2}\theta}$ 

For using both double angle formulae M1 correctly

 $= i \cot \frac{1}{2}\theta$ 

A1 3 For fully correct proof to AG

(ii)



M1 For a circle centre O

**A**1 For indication of radius = 1and anticlockwise arrow shown

В1 3 For locus of w shown as imaginary axis described downwards

6

3 (i)	METHOD 1	M1	For correct auxiliary equation (soi)
	$m+4 (=0) \Rightarrow CF (y=)Ae^{-4x}$	A1 2	For correct CF
	METHOD 2		
	Separating variables on $\frac{dy}{dx} + 4y = 0$		
	$\Rightarrow \ln y = -4x$	M1	For integration to this stage
	$\Rightarrow$ CF $(y =)Ae^{-4x}$	A1	For correct CF
(ii)	$PI(y =) p \cos 3x + q \sin 3x$	B1	For stating PI of correct form
	$y' = -3p\sin 3x + 3q\cos 3x$	M1	For substituting $y$ and $y'$ into DE
	$\Rightarrow (-3p+4q)\sin 3x + (4p+3q)\cos 3x = 5\cos 3x$	A1	For correct equation
		M1	For equating coeffs and solving
	$\Rightarrow \frac{-3p+4q=0}{4p+3q=5} \Rightarrow p=\frac{4}{5}, q=\frac{3}{5}$	A1 A1	For correct value of $p$ , and of $q$
	GS $(y =) Ae^{-4x} + \frac{4}{5}\cos 3x + \frac{3}{5}\sin 3x$	B1√ <b>7</b>	For GS
	$\frac{1}{5}\cos 3x + \frac{1}{5}\sin 3x$		f.t. from their CF+PI with 1 arbitrary constant
	SD Integrating factor method may be use	ad fallawa	in CF and none in PI and by 2-stage integration by parts or C+iS method
	SK integrating factor method may be use		for (i) are awarded only if CF is clearly identified
(iii)	$e^{-4x} \to 0$ , $\frac{4}{5}\cos 3x + \frac{3}{5}\sin 3x = \frac{\sin(3x + \alpha)}{\cos(3x + \alpha)}$	M1	For considering either term
	3 000	A1√ <b>2</b>	For correct range (allow < ) <b>CWO</b>
	$\Rightarrow -1 \leqslant y \leqslant 1  OR  -1 \lessapprox y \lessapprox 1$		f.t. as $-\sqrt{p^2 + q^2} \le y \le \sqrt{p^2 + q^2}$ from (ii)
		11	$\frac{1.1. \text{ as}  \sqrt{p} + q  \text{if our } (\mathbf{n})}{2}$
		11	
4 (i)	abc = (ab)c = (ba)c = b(ac) =	M1	For using commutativity correctly
	b(ca) = (bc)a = (cb)a = cba	A1 2	For correct proof (use of associativity may be implied)
	Minimum working:		(use of associativity may be implied)
	$abc = bac = bca = cba$ $OR \ abc = acb = cab = cba$		
	$OR \ abc = acb = cab = cba$ $OR \ abc = bac = bca = cba$		
(ii)	$\{e, a\}, \{e, b\}, \{e, c\}, \{e, bc\}, \{e, ca\}, \{e, ab\}, \{e, abc\}$	 B1	For any 5 subgroups
		B1 2	For the other 2 subgroups and none incorrect
(iii)	$\{e, a, b, ab\}, \{e, a, c, ca\}, \{e, b, c, bc\}$	B1	For any 3 subgroups
	$\{e, a, bc, abc\}, \{e, b, ca, abc\}, \{e, c, ab, abc\}$	B1	For 1 more subgroup
	$\{e, bc, ca, ab\}$	B1 <b>3</b>	For 1 more subgroup (5 in total)
			and nana incorract
(iv)	• • • • • • • • • • • • • • • • • • • •	B1*	and none incorrect  For appropriate reference to order of elements
(iv)	All elements $(\neq e)$ have order 2	B1*	
(iv)	All elements $(\neq e)$ have order 2  OR all are self-inverse  OR no element of G has order 4	B1*	For appropriate reference to order of elements
(iv)	All elements $(\neq e)$ have order 2  OR all are self-inverse  OR no element of G has order 4  OR no order 4 subgroup has a generator or is cyclic	B1*	For appropriate reference to order of elements
(iv)	All elements $(\neq e)$ have order 2  OR all are self-inverse  OR no element of G has order 4  OR no order 4 subgroup has a generator or is cyclic  OR subgroups are of the form $\{e, a, b, ab\}$	B1*	For appropriate reference to order of elements
(iv)	All elements $(\neq e)$ have order 2  OR all are self-inverse  OR no element of G has order 4  OR no order 4 subgroup has a generator or is cyclic  OR subgroups are of the form $\{e, a, b, ab\}$ (the Klein group)		For appropriate reference to order of elements in $G$
(iv)	All elements $(\neq e)$ have order 2  OR all are self-inverse  OR no element of G has order 4  OR no order 4 subgroup has a generator or is cyclic  OR subgroups are of the form $\{e, a, b, ab\}$	B1*  B1 (*dep)2	For appropriate reference to order of elements

5 (i)	$\frac{dy}{dx} = k u^{k-1} \frac{du}{dx}$	M1	For using chain rule
	$\frac{dx}{dx} - \kappa u = \frac{dx}{dx}$	A1	For correct $\frac{dy}{dx}$
	1		•
	$\Rightarrow xku^{k-1}\frac{\mathrm{d}u}{\mathrm{d}x} + 3u^k = x^2u^{2k}$	M1	For substituting for y and $\frac{dy}{dx}$
	$\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{3}{kx}u = \frac{1}{k}xu^{k+1}$	A1 <b>4</b>	For correct equation AG
(ii)	<i>k</i> = −1	B1 <b>1</b>	For correct k
(iii)	$\frac{du}{dx} - \frac{3}{x}u = -x \implies \text{IF}  e^{-\int \frac{3}{x} dx} = e^{-3\ln x} = \frac{1}{x^3}$	в1√	For correct IF
	$dx  x \qquad \exists  \mathbf{n}  \mathbf{c} \qquad \exists  \mathbf{c} \qquad \mathbf{c} \qquad \mathbf{c}$		f.t. for IF = $x^{\frac{3}{k}}$
			using $k$ or their numerical value for $k$
	d(1)	3.61	- ·
	$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x} \left( u \cdot \frac{1}{x^3} \right) = -\frac{1}{x^2}$	M1	For $\frac{d}{dx}(u)$ their IF = $-x$ their IF
	$\rightarrow v^{-1} - 1$ (10) $\rightarrow v^{-1}$	A1	For correct integration both sides
	$\Rightarrow u \cdot \frac{1}{x^3} = \frac{1}{x} (+c) \Rightarrow y = \frac{1}{cx^3 + x^2}$	A1 <b>4</b>	For correct solution for <i>y</i>
		9	
6 (a)	Closure $(ax+b)+(cx+d) = (a+c)x+(b+d)$	B1	For obtaining correct sum from 2 distinct
	n.	D.1	elements
	$\in P$	B1	For stating result is in <i>P</i> OR is of the correct form
			SR award this mark if any of the closure
			result, the identity or the inverse element is
			stated to be in <i>P OR</i> of the correct form
	Identity $0x + 0$	B1	For stating identity (allow 0)
	Inverse $-ax-b$	B1 <b>4</b>	For stating inverse
(b) (i)	Order 9	B1* <b>1</b>	For correct order
(ii)	<i>x</i> + 2	B1 <b>1</b>	For correct inverse element
(iii)	( , 1) , ( , 1) , ( , 1) 2 , 21	M1	For considering sums of $ax + b$
	(ax+b)+(ax+b)+(ax+b) = 3ax+3b		and obtaining $3ax + 3b$
	=0x+0		For equating to $0x + 0$ $OR$ 0
	$\Rightarrow ax + b$ has order $3 \forall a, b$ (except $a = b = 0$ )	A1	and obtaining order 3
	· · · · · ·		<b>SR</b> For order 3 stated only <i>OR</i> found from
			incomplete consideration of numerical cases award B1
	Cyclic group of order 9 has element(s) of order 9	M1	For reference to element(s) of order 9
		(*dep)	
	$\Rightarrow (Q, + (\text{mod } 3))$ is not cyclic	A1 4	For correct conclusion
		10	
-			

7 (i)	R Q	В1	For sketch of tetrahedron labelled in some way At least one right angle at <i>O</i> must be indicated or clearly implied
	o P	M1	For using $\Delta = \frac{1}{2}$ base × height
	$\Delta OPQ = \frac{1}{2} pq$ , $\Delta OQR = \frac{1}{2} qr$ , $\Delta ORP = \frac{1}{2} rp$	A1 3	For all areas correct CAO
(ii)	$\frac{1}{2} \left  \overrightarrow{RP} \times \overrightarrow{RQ} \right  = \frac{1}{2} \left  \overrightarrow{RP} \right  \left  \overrightarrow{RQ} \right  \sin R = \Delta PQR$	B1 <b>1</b>	For correct justification
(iii)	LHS = $\left(\frac{1}{2}pq\right)^{2} + \left(\frac{1}{2}qr\right)^{2} + \left(\frac{1}{2}rp\right)^{2}$	B1	For correct expression
	$\Delta PQR = \frac{1}{2}  (p\mathbf{i} - q\mathbf{j}) \times (p\mathbf{i} - r\mathbf{k}) $	B1	For $\triangle PQR$ in vector form
	$OR  \frac{1}{2}  (p\mathbf{i} - r\mathbf{k}) \times (q\mathbf{j} - r\mathbf{k}) $		
	$OR  \frac{1}{2}  (p\mathbf{i} - q\mathbf{j}) \times (q\mathbf{j} - r\mathbf{k}) $		
	$\Delta PQR = \frac{1}{2}  qr\mathbf{i} + pr\mathbf{j} + pq\mathbf{k} $	M1	For finding vector product of their attempt at $\Delta POR$
		A1	For correct expression
	RHS = $\frac{1}{4} \left( (pq)^2 + (qr)^2 + (rp)^2 \right)$	M1	For using $ a\mathbf{i} + b\mathbf{j} + c\mathbf{k}  = \sqrt{a^2 + b^2 + c^2}$
	,	A1 6	For completing proof of <b>AG WWW</b>
		10	

8 (i)			For expanding $(c+is)^4$ : at least 2 terms and
	$Re(c+is)^4 = \cos 4\theta = c^4 - 6c^2s^2 + s^4$	M1*	1 binomial coefficient needed
	4 2	A1 M1	For 3 correct terms
	$\cos 4\theta = c^4 - 6c^2(1 - c^2) + (1 - c^2)^2$	(*dep)	For using $s^2 = 1 - c^2$
	$\Rightarrow \cos 4\theta = 8\cos^4 \theta - 8\cos^2 \theta + 1$	A1 <b>4</b>	For correct expression for $\cos 4\theta$ <b>CAO</b>
(ii)	$\cos 4\theta \cos 2\theta = (8c^4 - 8c^2 + 1)(2c^2 - 1)$		For multiplying by $(2c^2-1)$
	$=16\cos^{6}\theta - 24\cos^{4}\theta + 10\cos^{2}\theta - 1$	B1 <b>1</b>	to obtain AG WWW
(iii)	$16c^6 - 24c^4 + 10c^2 - 2 = 0$	M1	For factorising sextic
	$\Rightarrow (c^2 - 1)(8c^4 - 4c^2 + 1) = 0$		with $(c-1)$ , $(c+1)$ or $(c^2-1)$
	For quartic, $b^2 - 4ac = 16 - 32 < 0$	A1	For justifying no other roots CWO
	$\Rightarrow c = \pm 1 \text{ only } \Rightarrow \theta = n \pi$	A1 <b>3</b>	For obtaining $\theta = n \pi$ <b>AG</b>
			Note that M1 A0 A1 is possible
		SR	For verifying $\theta = n \pi$ by substituting $c = \pm 1$
			into $16c^6 - 24c^4 + 10c^2 - 2 = 0$ B1
(iv)	$16c^6 - 24c^4 + 10c^2 = 0$		
	$\Rightarrow c^2 \left( 8c^4 - 12c^2 + 5 \right) = 0$	M1	For factorising sextic with $c^2$
	For quartic, $b^2 - 4ac = 144 - 160 < 0$	A1	For justifying no other roots CWO
	$\Rightarrow \cos \theta = 0$ only	A1 <b>3</b>	For correct condition obtained AG
			Note that M1 A0 A1 is possible
		SR	For verifying $\cos \theta = 0$ by substituting $c = 0$
			into $16c^6 - 24c^4 + 10c^2 = 0$ B1
		SR	For verifying $\theta = \frac{1}{2}\pi$ and $\theta = -\frac{1}{2}\pi$ satisfy
			$\cos 4\theta \cos 2\theta = -1$ B1
		11	

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